

HOPF RIGIDITY FOR CONVEX BILLIARDS ON THE HEMISPHERE AND HYPERBOLIC PLANE

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ABSTRACT. This paper deals with Hopf type rigidity for convex billiards on surfaces of constant curvature. We prove that the only convex billiard without conjugate points on the Hyperbolic plane or on the Hemisphere is circular billiard.

1. INTRODUCTION AND THE RESULT

In this paper we consider convex billiard with smooth (class C^2) boundary curve γ which lies on a constant curvature $K = \pm 1$ surface, denoted by S , hemisphere or hyperbolic plane. We shall assume that geodesic curvature k of the boundary curve γ everywhere positive.

Inspired by a famous theorem of E.Hopf [10] (see [5] for the generalization), I proved for the Euclidean plane in [2] that the only billiard which has no conjugate points (as a discrete dynamical system) is circular billiard. This is in fact very geometric result because as corollary one gets that the only *totally* integrable billiard in the plane is circular billiard (total integrability, after [14], means that the phase cylinder is foliated by closed invariant curves which are not null homotopic). The proof of this result consists of two steps (very much like in the Hopf theorem): the first is the construction of non-vanishing Jacobi field along every orbit, and the second is an argument of integral geometry.

It was observed by M. Wojtkowski in [17] that this result can be proved by a different integral geometric argument which uses the "mirror" formula of geometric optics. We refer to [18], [8] for the discussion and open problems and to [12] to other developments. However it was an old question which people tried to answer if this Hopf rigidity still holds for other models of billiards, which also give rise to twist symplectic maps of the cylinder. For all of these models the first part of the proof constructing non-vanishing Jacobi fields can be also performed. But integral geometric part is not known and not clear.

In this paper we prove the rigidity result for convex billiards on surfaces of constant, positive or negative curvature. The idea is again integral geometric and uses the "mirror formula" in the case of constant curvature. Using this tool we reduce the claim to a geometric inequality which turns out to be a consequence of the isoperimetric inequality on the surface. However, in the

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case of constant non-zero curvature this reduction and the reduced geometric inequality become much more involved, just because the isoperimetric inequalities on surfaces are essentially more complicated. Remarkably in the case of sphere the reduced geometric inequality can be obtained from Fenchel inequality for space closed curves, but I am not aware of such a proof for Hyperbolic case. For both cases we give the intrinsic proofs in Sections 5,6 below.

Let me remind the definition of conjugate points for billiard configurations (see [2]) or more generally for a twist map. Throughout this paper we choose an arc-length parameter x on γ and identify x with the $\gamma(x)$ everywhere below. Recall that billiard configurations $\{x_n\}$ are extremals of the action functional

$$(1) \quad A\{x_n\} = \sum_{-\infty}^{\infty} L(x_n, x_{n+1}),$$

where $L(x, y) = \text{dist}_S(x, y)$ is the generating function of the billiard ball map:

$$T : (x, \Phi) \rightarrow (y, \Psi), \quad \Phi = -L_1(x, y), \quad \Psi = L_2(x, y).$$

Here $\Phi = \cos \phi$, $\Psi = \cos \psi$ and ϕ, ψ are the angles formed by the oriented geodesic segment g_{xy} joining x and y with the oriented boundary curve γ . Subindexes of L here and later stand for partial derivatives. Then it follows that twist condition (see below) is satisfied and T is a symplectic twist map of the cylinder $\Omega = \gamma \times (-1, 1)$. A Jacobi field along configuration x_n is a sequence ξ_n satisfying the discrete Jacobi equation:

$$(2) \quad b_{n-1}\xi_{n-1} + a_n\xi_n + b_n\xi_{n+1} = 0,$$

$$a_n = L_{22}(x_{n-1}, x_n) + L_{11}(x_n, x_{n+1}), \quad b_n = L_{12}(x_n, x_{n+1}).$$

Two points x_M, x_N of the configuration are called conjugate if there exists a non-trivial Jacobi field vanishing at these points.

Our main result is as follows:

Theorem 1.1. *Let γ be a smooth convex simple closed curve having positive geodesic curvature lying on the hemisphere or on the hyperbolic plane of constant curvature ± 1 . If every billiard configuration $\{x_n\}$ has no conjugate points, then γ is a geodesic circle.*

Corollary 1.2. *Suppose the billiard ball map T is totally integrable, meaning that through every point of the phase cylinder passes a continuous non-contractible curve invariant under T . Then γ is a geodesic circle.*

Three remarks are in order:

1. The model of billiards on constant curvature surfaces (and even more general models) is extensively studied. We refer to [6], [19], [7], [16] which are most relevant to our result.

It is worth mentioning that similarly to the planar case the elliptic domains on the hemisphere and hyperbolic plane are the only known examples of integrable billiards ([16]). Corollary 1.2 proves a part of the Birkhoff conjecture on constant curvature surfaces saying that the only integrable convex billiards are ellipses. We refer to [13] for other developments in the direction of this conjecture.

2. In [17] Wojtkowski uses the assumption of existence of the so-called monotone invariant sub-bundle replacing the assumption of no conjugate points. It is a consequence of [2] that these two assumptions are in fact equivalent as we shall explain in Section 3. However I prefer my original condition of no conjugate points.

3. It is a remarkable fact that for billiard in any convex domain of Euclidean space of dimensions greater than 2 there always exist conjugate points (see [3]). This is in fact a very general phenomenon valid for Riemannian billiards inside simple convex domains, starting from dimension 3. It is still unclear to me how billiard inside the round ball can be distinguished in terms of variational properties of billiard orbits.

This paper is organized as follows. I shall discuss equivalent forms of the "no conjugate points" condition for twist maps in Section 3 below. The proof of the main theorem uses "mirror" equation. We show how one gets the "mirror" equation from our assumption of no conjugate points in Section 4. The main integral geometric part of the proof is done separately: first for the hyperbolic plane in Section 5 and then for the hemisphere in Section 6.

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2. MIRROR FORMULA

The mirror formula of geometric optics for a convex mirror in the Euclidean plane reads

$$\frac{1}{a} + \frac{1}{b} = \frac{2k(x)}{\sin \phi}.$$

Here x is a point on the mirror, a is a distance from a point A inside the domain to the point x along the shortest ray and b is a distance from x along the reflected ray to the point B where the focusing of the reflected beam occurs, ϕ is the angle of reflection. The mirror formula can be generalized to simple convex domains in Riemannian surfaces in a straightforward way. (Being simple means that all geodesics inside the domain are minimal, intersecting the boundary curve, in particular there are no conjugate points along geodesics inside the domain). In Riemannian case the mirror formula is the following:

$$\frac{Y'_A(a)}{Y_A(a)} + \frac{Y'_B(b)}{Y_B(b)} = \frac{2k(x)}{\sin \phi}.$$

Here the Y denotes the orthogonal Jacobi field along geodesic segments $g_{A,x}$ and $g_{B,x}$ going to x from A and B respectively. In this paper we shall assume everywhere that subindex A at Y means that A is the initial point on geodesic segment the initial conditions for Jacobi equation are.

$$Y_A(0) = 0; Y'_A(0) = 1.$$

It is important that for the case of constant curvature the Jacobi field with these initial conditions do not depend on the point and the direction and are equal to

$$Y(t) = \sinh t; \quad Y(t) = \sin t,$$

for the case of hyperbolic plane and the hemisphere respectively. So in this case the formula looks particularly simple (we refer to [6], [19] for some billiard applications of this formula in constant curvature case, and to [7] for mirror formula for other billiards):

$$\frac{Y'}{Y}(a) + \frac{Y'}{Y}(b) = \frac{2k(x)}{\sin \phi}.$$

The proof of the mirror formula in general Riemannian case follows immediately from the explicit computation of the second derivatives of the length function L (the calculations using Gauss Bonnet theorem are easy and will not be reproduced here):

$$\begin{aligned} L_{11}(x, y) &= \frac{Y'_y}{Y_y}(L) \sin^2 \phi - k(x) \sin \phi, \\ (3) \quad L_{22}(x, y) &= \frac{Y'_x}{Y_x}(L) \sin^2 \psi - k(y) \sin \psi, \\ L_{12}(x, y) &= \frac{\sin \phi \sin \psi}{Y_x(L)} = \frac{\sin \phi \sin \psi}{Y_y(L)}. \end{aligned}$$

Here Y_x and Y_y are Jacobi fields along the segments g_{xy} and g_{yx} with initial conditions at the points x and y respectively. Again in the case of constant curvature the subindexes can be dropped.

3. THE CONDITION OF NO CONJUGATE POINTS

In this section T denotes symplectic negative twist map of the cylinder (not necessarily billiard map), with the generating function L . Let me remind the condition used in [17] instead of no conjugate points condition.

Let l be a *non-vertical* sub-bundle of tangent lines to the cylinder Ω . In other words the line $l(x, \Phi) \subset T_{(x, \Phi)}\Omega$ is assumed to be non-vertical for all $(x, \Phi) \in \Omega$. Such a sub-bundle is called monotone in [17] if in addition it is equipped with the orientation chosen on the lines by the condition $dx > 0$.

In order to state all the equivalent conditions, I shall use also the definition of locally maximizing configurations of negative twist maps (note the positive sign of mixed derivative of L in the case of billiards). Infinite extremal $\{x_n\}$ is called locally maximizing if any finite segment of the extremal is a local maximum of the functional with the fixed end points.

Theorem 3.1. *Given a twist area preserving map T of the cylinder Ω with the generating function L (with the positive cross derivative L_{12} , the so called negative twist). The following conditions are equivalent:*

- (a) *All configurations have no conjugate points;*
- (b) *All configurations are locally maximizing;*

- (c) *There exists a measurable positive function $\nu : \Omega \rightarrow \mathbf{R}_+$ such that the cocycle*

$$\nu_n(x, \Phi) = \nu(x, \Phi) \cdot \dots \cdot \nu(T^{n-1}(x, \Phi))$$

satisfies the discrete Jacobi equation (2).

- (d) *There exists measurable non-vertical, monotone sub-bundle defined a.e. on Ω which is invariant under the twist map T .*

It is a well known fact in Aubry-Mather theory that any configuration $\{x_n\}$ which comes from an orbit lying on a closed continuous non-contractible invariant curve of T is locally maximizing. This fact implies that if one assumes the "foliation condition", meaning that there exists an invariant curve passing through every point of the cylinder, then each of the conditions (a)-(d) of Theorem 3.1 follows. This remark explains Corollary 1.2 of the main theorem.

Let me remark also that it is plausible that this "foliation condition" is in fact equivalent to the conditions (a)-(d), like it is established ([11] see also [1]) for the case of the flows. However, I don't know the proof of this statement for the case of twist maps.

Proof. Condition (b) is equivalent to (a) for the following reason. The matrix of the second variation of any locally maximizing segment is negative semi-definite. Then it follows from [15] that in such a case it is in fact negative definite. The condition of "no conjugate points" along one configuration is equivalent to the fact that for any segment the matrix of second variation is non-degenerate. Therefore there are no conjugate points along every locally maximizing orbit, so (b) implies (a). In the opposite direction, if all configurations are known to have no conjugate points then second variation matrix of any finite segment of any orbit is non-degenerate and therefore must be positive definite by a simple continuity argument, since among all configurations there are Aubry-Mather configurations having negative definite second variation.

Let me explain that (a) implies (c). This implication is proved in [2] generalizing the limiting argument by Hopf. For any configuration $\{x_n\}$ with no conjugate points one constructs stable positive Jacobi field $\{\nu_n\}$, $\nu_0 = 1$ along it. These ν_n are measurable positive functions on Ω which form a cocycle satisfying discrete Jacobi equation (2).

The implication (c) \Rightarrow (d) is obvious. Just define the monotone sub-bundle consisting of oriented tangent lines $l(x, \Phi)$ in $T_{(x, \Phi)}\Omega$ spanned by the vectors

$$\frac{\partial}{\partial x} - (L_{11}(x, y) + L_{12}(x, y)\nu_1(x, \Phi))\frac{\partial}{\partial \Phi}.$$

It is measurable, monotone, and invariant under T . The invariance follows easily from the discrete Jacobi equation (2). The last implication we need to prove is (d) \Rightarrow (a). For this we use the following:

Lemma 3.2. *Let $\{(x_n, \Phi_n), n = 0, \dots, N\}$ be a finite segment of an orbit of T , $(x_n, \Phi_n) = T^n(x_0, \Phi_0), n = 0, \dots, N$. Let us assume that $\{w_n, n = 0, \dots, N\}$ is a field of tangent vectors to Ω at the points (x_n, Φ_n) such that $w_n = DT^n(w_0)$ with the "monotonicity" property, that is $dx(w_n) > 0$ for all $0 \leq n \leq N$. Then any Jacobi field along the segment has at most one "generalized" zero*

(or sign change, see [9]), in particular the points x_i and x_k are not conjugate for any $0 \leq i < k \leq N$.

Proof of Lemma. Given w_n like in lemma. Projection of the invariant field w_n defines a *positive* Jacobi field $\{\xi_0, \dots, \xi_n\}$ along the configuration $\{x_0, \dots, x_n\}$. Then it follows from discrete version of Sturm Separation theorem ([9], theorem 7.9) that no Jacobi field can change sign along the segment and in particular no conjugate points can occur. \square

It follows from the lemma that all orbits lying in the support of the monotone sub-bundle do not have conjugate points. The support is a set of full measure. Then I claim that all orbits in the closure of the support (i.e. all the orbits) have no conjugate points as well. Indeed, take any finite segment $\{x_0, \dots, x_{N+1}\}$ of an orbit lying in the support of the monotone sub-bundle. Then one can find a unique Jacobi field $\{1 = \xi_0, \dots, \xi_{N+1} = 0\}$. It is possible since x_0, x_{N+1} are not conjugate. Then it follows from the lemma $\xi_i, i = 1, \dots, N$ must be positive.

Moreover having a sequence of such segments $\{x_0^{(k)}, \dots, x_{N+1}^{(k)}\}$ which converges to a segment $\{y_0, \dots, y_{N+1}\}$ when $k \rightarrow \infty$ one can pass to the limit and thus to get a non-negative Jacobi field $\{1 = \zeta_0, \dots, \zeta_N, \zeta_{N+1} = 0\}$ along the limiting segment $\{y_0, \dots, y_{N+1}\}$ with the property $\zeta_i \geq 0$. Then it is clear that ζ_i are in fact positive for all $i = 0, \dots, N$. So by the lemma the segment $\{y_0, \dots, y_N\}$ has no conjugate points. Thus all orbits have no conjugate points and so (a) follows. This completes the proof of the theorem. \square

Remark 1. It is impossible to drop the monotonicity condition in the assumption (d) of the last theorem. For example, one can take the phase portrait of the elliptical billiard in the plane and remove all the iterates of the zero section of the phase cylinder in order to get invariant non-vertical sub-bundle defined on a set of full measure. This sub-bundle is not monotone, and of course there are conjugate points for elliptical billiard.

4. MIRROR EQUATION

We state and prove in this section the mirror equation on a measurable function $a : \Omega \rightarrow \mathbf{R}$. It follows from the mirror formula and no conjugate points condition. It could be proved for general case of billiards in simple Riemannian domains. But in order to simplify the notations we shall do it only for constant curvature case. General case is absolutely analogous with special care taken on the initial points of the Jacobi fields involved.

Theorem 4.1. *If the billiard has no conjugate points, then there exists a measurable function on the phase cylinder $a : \Omega \rightarrow \mathbf{R}$ such that $0 < a(x, \Phi) < L(x, \Phi)$ which satisfies the mirror equation:*

$$(4) \quad \frac{Y'}{Y}(a(x, \Phi)) + \frac{Y'}{Y}(L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1})) = \frac{2k(x)}{\sin \phi}$$

For the Euclidean case this theorem was proved in [17] under the assumption (d) of the existence of invariant monotone sub-bundle of the billiard map. I shall show how to construct geometrically this function assuming the condition (a) of no conjugate points. The geometric meaning of this

function is the distance traveled along the geodesic to the point lying on the caustic. This is defined as follows.

Let $\{x_n\}$ be a billiard configuration with no conjugate points. Take the geodesic segment g_{x_0, x_1} . Since this geodesic has no conjugate points there exists a unique orthogonal Jacobi field J along this geodesic which has the following boundary values at the ends of the segment

$$J(x_0) = \nu_0 \sin \phi_0; \quad J(x_1) = -\nu_1(x_0, \Phi_0) \sin \phi_1.$$

Here $\nu_0 = 1$ and ν_1 is positive function of the cocycle ν_n described in condition (c) of Theorem 3.1, that is $\nu_0 = 1$ and ν_1 are the values of the positive Jacobi field at the points x_0, x_1 respectively. Since the signs of J at the ends of the segment are opposite then there exists a unique point z between x_0 and x_1 where J vanishes. So we define $a(x_0, \Phi_0)$ to be the distance along the segment from x_0 to z . One can easily see that $a(x_0, \Phi_0)$ must be the unique solution of the equation

$$(5) \quad \frac{Y(a)}{Y(L(x_0, \Phi_0) - a)} = \frac{\sin \phi_0}{\nu_1(x_0, \Phi_0) \sin \phi_1}$$

It can be easily seen that the function on the left hand side defined on $(0, L)$ is strictly monotone increasing (one can compute the derivative using Wronskian) growing from 0 at 0 to $+\infty$ at L . Therefore there is a unique solution depending continuously both on L and on the right hand side. Since the cocycle is measurable then the function a is measurable also. Analogously to $a(x_0, \Phi_0)$ we have

$$(6) \quad \begin{aligned} \frac{Y(a(x_{-1}, \Phi_{-1}))}{Y(L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1}))} &= \frac{\sin \phi_{-1}}{\nu_1(x_{-1}, \Phi_{-1}) \sin \phi_0} = \\ &= \frac{\sin \phi_{-1} \nu_{-1}(x_0, \Phi_0)}{\sin \phi_0}, \end{aligned}$$

where the last equality in (6) follows from the cocycle property.

The last thing is to show how the mirror equation (4) follows. It turns out that it is just another form of the Jacobi equation (7):

$$(7) \quad L_{12}(x_{-1}, x_0) \nu_{-1} + (L_{22}(x_{-1}, x_0) + L_{11}(x_0, x_1)) + L_{12}(x_0, x_1) \nu_1 = 0.$$

Indeed, substituting the explicit formulas for the second derivatives (3) of L and dividing by $\sin^2 \phi_0$ one can rewrite (7) in the form

$$(8) \quad \begin{aligned} &\frac{\nu_{-1}(x_0, \Phi_0) \sin \phi_{-1}}{Y(L(x_{-1}, \Phi_{-1})) \sin \phi_0} + \frac{Y'}{Y}(L(x_{-1}, \Phi_{-1})) + \\ &+ \frac{Y'}{Y}(L(x_0, \Phi_0)) + \frac{\nu_1(x_0, \Phi_0) \sin \phi_1}{Y(L(x_0, \Phi_0)) \sin \phi_0} = \frac{2k(x_0)}{\sin \phi_0}. \end{aligned}$$

We can rewrite the first and the last expression in (8) by (6) and (5) respectively to get

$$(9) \quad \frac{Y(a(x_{-1}, \Phi_{-1}))}{Y(L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1}))Y(L(x_{-1}, \Phi_{-1}))} + \frac{Y'}{Y}(L(x_{-1}, \Phi_{-1})) + \frac{Y'}{Y}(L(x_0, \Phi_0)) + \frac{Y(L(x_0, \Phi_0) - a(x_0, \Phi_0))}{Y(L(x_0, \Phi_0))Y(a(x_0, \Phi_0))} = \frac{2k(x_0)}{\sin \phi_0}.$$

Now using the fact that Wronskian of any two Jacobi fields is constant we can write the identities:

$$Y(a) = Y(L)Y'(L - a) - Y'(L)Y(L - a)$$

and

$$Y(L - a) = Y(L)Y'(a) - Y(a)Y'(L),$$

(in the constant curvature case these are just usual trigonometric identities). Use these identities on the left hand side of (9), the first identity-for the nominator of the first summand and the second identity-for the nominator of the last summand. Then it can be easily verified that this yields exactly the mirror equation (4) claimed in the theorem.

5. PROOF OF RIGIDITY FOR HYPERBOLIC PLANE

In this section we prove the main theorem for billiards on the hyperbolic plane.

Write the mirror equation for the Hyperbolic plane (4):

$$(10) \quad \coth(a(x, \Phi)) + \coth(L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1})) = \frac{2k(x)}{\sin \phi}$$

Recall that for $t > 0$, $\coth t$ is a convex function on t with $\coth t > 1$. Since the equation (10) holds true for every (x, Φ) we can take $\phi = \pi/2$ and thus immediately get that

$$k(x) > 1$$

for any x . In other words it follows that our domain must be convex with respect to horocycles on the Hyperbolic plane. Moreover using convexity one has

$$\coth \frac{a(x, \Phi) + L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1})}{2} \leq \frac{k(x)}{\sin \phi}.$$

This can be written in the equivalent form

$$\frac{a(x, \Phi) + L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1})}{2} \geq \operatorname{arctanh} \left(\frac{\sin \phi}{k(x)} \right)$$

Integrate the last inequality with respect to the invariant measure $d\mu = dx \, d\Phi = \sin \phi \, dx \, d\phi$ to get

$$\begin{aligned} \int L \, d\mu &\geq 2 \int_0^P dx \int_0^\pi \operatorname{arctanh} \left(\frac{\sin \phi}{k(x)} \right) \sin \phi \, d\phi = \\ &= 4 \int_0^P dx \int_0^{\pi/2} \operatorname{arctanh} \left(\frac{\sin \phi}{k(x)} \right) \sin \phi \, d\phi. \end{aligned}$$

Here P is the perimeter of the boundary curve γ . For every x compute the inner integral on the right hand side integrating by parts

$$\begin{aligned} \int_0^{\pi/2} \operatorname{arctanh}\left(\frac{\sin \phi}{k(x)}\right) \sin \phi \, d\phi &= k(x) \int_0^{\pi/2} \frac{\cos^2 \phi}{k^2(x) - \sin^2 \phi} \, d\phi = \\ &= \frac{\pi}{2}(k(x) - \sqrt{k^2(x) - 1}). \end{aligned}$$

Using Santalo' formula $\int L \, d\mu = 2\pi A$ (A is the area of the domain) we obtain the following inequality

$$A \geq \int_0^P (k(x) - \sqrt{k^2(x) - 1}) \, dx.$$

Use Gauss-Bonnet to write it in the form

$$A \geq 2\pi + A - \int_0^P \sqrt{k^2(x) - 1} \, dx$$

and therefore

$$\int_0^P \sqrt{k^2(x) - 1} \, dx \geq 2\pi$$

But then it follows from the next lemma stating the opposite inequality that the curve γ must be a circle.

Lemma 5.1. *For any simple closed curve γ on the Hyperbolic plane which is convex with respect to horocycles the following inequality holds true*

$$\int_0^P \sqrt{k^2(x) - 1} \, dx \leq 2\pi,$$

where the equality is possible only for circles.

Proof. The integral can be estimated from above by the Cauchy-Schwartz

$$\begin{aligned} \int_0^P \sqrt{k^2(x) - 1} \, dx &\leq \left(\int_0^P (k(x) - 1) \, dx \right)^{\frac{1}{2}} \left(\int_0^P (k(x) + 1) \, dx \right)^{\frac{1}{2}} = \\ &= ((A + 2\pi - P)(A + 2\pi + P))^{\frac{1}{2}}, \end{aligned}$$

where we have applied Gauss Bonnet formula. The last expression gives

$$((A + 2\pi - P)(A + 2\pi + P))^{\frac{1}{2}} = (A^2 + 4\pi A - P^2 + 4\pi^2)^{\frac{1}{2}} \leq 2\pi,$$

since by the isoperimetric inequality on the Hyperbolic plane

$$A^2 + 4\pi A - P^2 \leq 0.$$

□

This completes the proof for the Hyperbolic plane.

6. PROOF OF RIGIDITY FOR HEMISPHERE

In this section we treat the case of the hemisphere.

Write the mirror equation (4) for the Hemisphere:

$$(11) \quad \cot(a(x, \Phi)) + \cot(L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1})) = \frac{2k(x)}{\sin \phi}$$

We shall need the following easy lemma.

Lemma 6.1. *The following inequality holds true*

$$\cot\left(\frac{a+b}{2}\right) \leq \frac{\cot a + \cot b}{2},$$

for all a, b in the range $(0; \pi)$ satisfying $\frac{a+b}{2} \leq \pi/2$.

Proof. This is obvious in the case when both a and b belong to $(0; \pi/2]$ just by the convexity of \cot on this interval. In the remaining case one of the numbers, say a lies in $(0; \pi/2)$ and b in $(\pi/2; \pi)$. Since the average is $\leq \pi/2$ we can write $a = \pi/2 - x - \delta$ and $b = \pi/2 + x$, where x, δ are non-negative and $x + \delta < \pi/2$. With these notations one needs to prove that in this range,

$$\tan \frac{\delta}{2} \leq \frac{\tan(x + \delta) - \tan x}{2}.$$

Indeed we have by the trigonometric formula

$$\frac{\tan(x + \delta) - \tan x}{2} = \frac{(\tan \delta)(1 + \tan x \tan(x + \delta))}{2} \geq \frac{\tan \delta}{2} \geq \tan \frac{\delta}{2}.$$

This proves the lemma. \square

Now take the following two numbers

$$a = a(x, \Phi), \quad b = L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1}),$$

and notice that both of them lie in $(0; \pi)$ since the billiard domain lies entirely on the hemisphere. Also their average $(a + b)/2$ must be $\leq \pi/2$ since otherwise the sum $\cot a + \cot b$ would be negative which contradicts the equation (11). Thus Lemma 6.1 can be applied to get:

$$\cot \frac{a(x, \Phi) + L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1})}{2} \leq \frac{k(x)}{\sin \phi}.$$

This can be written in the equivalent form:

$$\frac{a(x, \Phi) + L(x_{-1}, \Phi_{-1}) - a(x_{-1}, \Phi_{-1})}{2} \geq \arctan\left(\frac{\sin \phi}{k(x)}\right).$$

Integrate the last inequality with respect to the invariant measure $d\mu = dx \, d\Phi = \sin \phi \, dx \, d\phi$ to get

$$\begin{aligned} \int L \, d\mu &\geq 2 \int_0^P dx \int_0^\pi \arctan\left(\frac{\sin \phi}{k(x)}\right) \sin \phi \, d\phi = \\ &= 4 \int_0^P dx \int_0^{\pi/2} \arctan\left(\frac{\sin \phi}{k(x)}\right) \sin \phi \, d\phi. \end{aligned}$$

Compute the inner integral on the right hand side integrating by parts

$$\begin{aligned} \int_0^{\pi/2} \arctan\left(\frac{\sin \phi}{k(x)}\right) \sin \phi \, d\phi &= k(x) \int_0^{\pi/2} \frac{\cos^2 \phi}{k^2(x) + \sin^2 \phi} \, d\phi = \\ &= \frac{\pi}{2} (\sqrt{k^2(x) + 1} - k(x)). \end{aligned}$$

Using Santalo' formula $\int L \, d\mu = 2\pi A$ again we obtain the following inequality

$$A \geq \int_0^P (\sqrt{k^2(x) + 1} - k(x)) \, dx.$$

Use Gauss-Bonnet to write it in the form

$$A \geq \int_0^P \sqrt{k^2(x) + 1} \, dx - (2\pi - A),$$

which leads to

$$\int_0^P \sqrt{k^2(x) + 1} \, dx \leq 2\pi.$$

But then the following lemma implies that the curve γ must be a circle thus completing the proof of the main theorem for the Hemisphere.

Lemma 6.2. *For any simple closed curve on the hemisphere the following inequality holds*

$$\int_0^P \sqrt{k^2(x) + 1} \, dx \geq 2\pi,$$

where the equality happens only for circles.

Remark 2. Notice that if the sphere is realized as a unite sphere in the Euclidean 3-space, then lemma follows from Fenchel inequality. This is because $\sqrt{k^2(x) + 1}$ equals to absolute curvature of the curve in Euclidean space. Next we give an intrinsic proof.

Proof. Denote by $I = \int_0^P \sqrt{k^2(x) + 1} \, dx$. We have by Cauchy- Schwartz and Gauss-Bonnet

$$\int_0^P (\sqrt{k^2(x) + 1} + 1) \, dx \cdot \int_0^P (\sqrt{k^2(x) + 1} - 1) \, dx \geq \left(\int_0^P k(x) \, dx \right)^2 = (2\pi - A)^2.$$

This can be rewritten as

$$(I - P)(I + P) \geq (2\pi - A)^2,$$

and hence

$$I^2 \geq P^2 + A^2 - 4\pi A + 4\pi^2 \geq 4\pi^2.$$

Here in the last inequality we used the isoperimetric inequality on the sphere:

$$P^2 + A^2 - 4\pi A \geq 0.$$

Thus $I \geq 2\pi$. The proof is completed. \square

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